EXTENSION DIMENSION AND QUASI-FINITE CW-COMPLEXES

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Abstract. We extend the definition of quasi-finite complexes by considering not necessarily countable complexes. We provide a characterization of quasi-finite complexes in terms of L-invertible maps and dimensional properties of compactifications. Several results related to the class of quasi-finite complexes are established, such as completion of metrizable spaces, existence of universal spaces and a version of the factorization theorem. Further, we extend the definition of $UV(L)$-spaces on non-compact case and show that some properties of $UV(n)$-spaces and $UV(n)$-maps remain valid, respectively, for $UV(L)$-spaces and $UV(L)$-maps.

1. Introduction

Extension theory introduced by Dranishnikov [14, 15] unifies the covering dimension and the cohomological dimension. There are two classes of maps which play an important role in extension theory. For a given complex L, theses are L-invertible and L-soft maps. It should be mentioned that universal spaces in dimension L as well as absolute extensors in dimension L are obtained as preimages of Hilbert cube or Hilbert space under maps from the above classes [10]. For a countable complex L, existence of L-invertible mapping of certain L-dimensional compactum onto the Hilbert cube is closely connected with the dimensional properties of compactifications of spaces with extension dimension not grater than L [9]. It turned out that the existence of such L-invertible mappings can be characterized in terms of "extensional" properties of a complex. This inspired the concept of quasi-finite countable complexes [20].

In the present paper we extend the definition of quasi-finite complexes by considering not necessarily countable complexes. We also provide a characterization of quasi-finite complexes in terms of L-invertible maps and dimensional properties of compactifications. Another interesting observation consists in the fact that many results established for finite or countable complexes remain valid

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for quasi-finite complexes. In particular, quasi-finite complexes possess the L-soft map property and every metrizable space of extension dimension $\leq L$ has a completion with the same extensional dimension. We also prove a version of the factorization theorem, and construct universal spaces. Finally, in case L being quasi-finite it is possible to define $UV(L)$-property for non-compact spaces. We show that this property does not depend on the embedding of a space into absolute neighborhood extensor in dimension L and obtain some results about $UV(L)$-maps and $UV(L)$-spaces which were known for $UV(n)$-maps and $UV(n)$-spaces, respectively.

2. Quasi-finite CW-complexes

Everywhere in this paper we assume that spaces are Tychonov and maps are continuous. Let X and Y be two spaces, $A \subset X$ and $g\colon A \to Y$ a map. We write $Y \in ANE(g, A, X)$ if g has a continuous extension $\bar{g}\colon U \to Y$, where U is a neighborhood of A in X which has the following property: there exists a function $h\colon X \to [0,1]$ such that $h^{-1}((0,1]) = U$ and $h(A) = 1$. If, in the above definition, $U = X$, we write $Y \in AE(g, A, X)$. Let us note that, by [16, Lemma 2.8], $Y \in ANE(g, A, X)$ if and only if g extends to a map $\overline{g}\colon X \to Cone(Y)$.

Everywhere below L always denotes a CW-complex.

We say that L is an absolute extensor of X, notation $L \in AE(X)$, if $L \in AE(g, A, X)$ for every closed $A \subset X$ and every map $g\colon A \to L$ with $L \in ANE(g, A, X)$. We say also that the extension dimension of X is not greater than L, notation e-dim$X \leq L$, if $L \in AE(X)$. Using Dydak's version of the Homotopy Extension Theorem [16, Theorem 13.7] one can show that if L_1 is homotopy equivalent to L_2, then e-dim$X \leq L_1$ is equivalent to e-dim$X \leq L_2$ for any space X. Moreover, our definition of e-dim coincides with that one of Chigogidze [8] in case L is countable and with the original definition of Dranishnikov [**?**] when compact spaces are considered.

A pair of spaces $K \subset P$ is called L-connected if whenever $A \subset X$ is a closed subset of a space X with e-dim$X \leq L$, then every map $g\colon A \to K$ has an extension $\overline{g}\colon X \to P$ provided A is normally placed in X with respect to (g, P). The notion of a normally placed set was introduced in [8] under different notation and means that for every continuous function h on P the function $h \circ g$ can be continuously extended over X. Obviously, this condition is satisfied for every normal space X and every map $g\colon A \to K$ with $A \subset X$ closed. We sometimes say that a pair $K \subset P$ is L-connected with respect to a given class of spaces $\mathcal{B}$ if the additional requirement $X \in \mathcal{B}$ is imposed in the above definition.

Quasi-finite CW-complexes were introduced in [20] as countable complexes L satisfying the following condition: every finite subcomplex K of L is contained in a finite subcomplex $P \subset L$ such that the pair $K \subset P$ is L-connected with respect to Polish spaces. It was also shown in [20] that there exists a countable

quasi-finite complex M extension type $[M]$ of which does not contain a finitely dominated complex (see [10] for more information on extension types). In this note we extend the above definition by considering not necessarily countable complexes. Here is our revised definition: a CW-complex L is quasi-finite if every finite subcomplex K of L is contained in a finite subcomplex $P \subset L$ such that the pair $K \subset P$ is L-connected. It is easy to verify that this definition coincides with the definition given in [20] in case L is countable.

We say that a map $f\colon X \to Y$ is L-invertible if for any map $g\colon Z \to Y$ with e-dim$Z \leq L$ there is a map $h\colon Z \to X$ such that $g = f \circ h$. If, in addition, Z is required to be from a given class of spaces $\mathcal{B}$, then we say that the map f is L-invertible with respect to the class $\mathcal{B}$. Everywhere below $w(X)$ denotes the weight of the space X and $\mathbb{I}^\tau$ denotes Tychonov cube of weight τ.

Theorem 2.1. *The following conditions are equivalent for any CW-complex L and an infinite cardinal τ:*

(1) *L is quasi-finite.*
(2) e-dim$\beta X \leq L$ *whenever X is a space with* e-dim$X \leq L$.
(3) *There exists an L-invertible map $f\colon Y_\tau \to \mathbb{I}^\tau$ such that Y_τ is a compact space of weight $\leq \tau$ and* e-dim$Y_\tau \leq L$.
(4) *For every L-connected pair $K \subset M$, where K is a compactum of weight $\leq \tau$ and M an arbitrary space, there exists a compactum $P \subset M$ containing K such that $w(P) \leq \tau$ and the pair $K \subset P$ is L-connected.*

Proof. (1) $\Rightarrow$ (2) Suppose e-dim$X \leq L$ and let $f\colon A \to L$, where A is a closed subset of βX. It is well known that every CW-complex is an absolute neighborhood extensor for the class of compact spaces, so $L \in ANE(f, A, \beta X)$ and there exists a closed neighborhood B of A in βX and a map $g\colon B \to L$ extending f. Because $g(B)$ is compact, it is contained in a finite subcomplex K of L. Since L is quasi-finite, there exists a finite subcomplex P of L such that the pair $K \subset P$ is L-connected. We can assume that B is a zero-set in βX. Then $B \cap X$, being a non-empty zero-set in X, is normally placed in X with respect to (g, P). Therefore, the map $g\colon B \cap X \to K$ extends to a map $h\colon X \to P$ because e-dim$X \leq L$ and the pair $K \subset P$ is L-connected. Finally, let $\overline{h}\colon \beta X \to P$ be the unique extension of h. Then $\overline{h}$ extends f, so e-dim$\beta X \leq L$.

(2) $\Rightarrow$ (3) We consider the family of all maps $\{h_\alpha\colon X_\alpha \to \mathbb{I}^\tau\}_{\alpha\in\Lambda}$ such that each X_α is a closed subset of $\mathbb{I}^\tau$ with e-dim$X_\alpha \leq L$. Let X be the disjoint sum of all X_α and the map $h\colon X \to \mathbb{I}^\tau$ coincides with h_α on every X_α. Clearly, e-dim$X \leq L$. Therefore, e-dim$\beta X \leq L$. Consider the extension $\overline{h}\colon \beta X \to \mathbb{I}^\tau$. Then, by the factorization theorem from [24], there exists a compact space Y_τ of weight $\leq \tau$ and maps $r\colon \beta X \to Y_\tau$ and $f\colon Y_\tau \to \mathbb{I}^\tau$ such that e-dim$Y_\tau \leq L$ and $f \circ r = \overline{h}$.

Let us show that f is L-invertible. Take a space Z with e-dim$Z \leq L$ and a map $g\colon Z \to \mathbb{I}^\tau$. Considering βZ and the extension $\overline{g}\colon \beta Z \to \mathbb{I}^\tau$ of g, we can

assume that Z is always compact. We also can assume that the weight of Z is $\leq \tau$ (otherwise we apply again the factorization theorem from [24] to find a compact space T of weight $\leq \tau$ and maps $g_1\colon Z \to T$ and $g_2\colon T \to \mathbb{I}^\tau$ with e-dim$T \leq L$ and $g_2 \circ g_1 = g$, and then consider the space T and the map g_2 instead, respectively, of Z and g). Therefore, without losing generality, we can assume that Z is a closed subset of $\mathbb{I}^\tau$. According to the definition of X and the map h, there is an index $\alpha \in \Lambda$ such that $Z = X_\alpha$ and $g = h_\alpha$. The restriction $r|Z\colon Z \to Y_\tau$ is a lifting of g, i.e. $f \circ (r|Z) = g$.

(3) $\Rightarrow$ (4) Suppose that K is a compact subset of the space M with $w(K) \leq \tau$ and $K \subset M$ being L-connected. We embed K in $\mathbb{I}^\tau$ and consider an L-invertible mapping $f\colon Y_\tau \to \mathbb{I}^\tau$ such that Y_τ is compact and e-dim$Y_\tau \leq L$. Let $\tilde{K} = f^{-1}(K)$ and $h = f|\tilde{K}$. Obviously, $\tilde{K}$ is normally placed in Y_τ with respect to (h, M). Consequently, h extends to a map $\overline{h}\colon Y_\tau \to M$ and let $P = \overline{h}(Y_\tau)$. Obviously, $w(P) \leq \tau$, so that it remains only to show that $K \subset P$ is L-connected. For this end, let $g\colon A \to K$, where $A \subset X$ is a closed normally placed subset of X with respect to (g, P) and e-dim$X \leq L$. This implies that A is normally placed in X with respect to $(g, \mathbb{I}^\tau)$. Since $\mathbb{I}^\tau$ is an absolute extensor, there exists an extension $g_1\colon X \to \mathbb{I}^\tau$ of g. Next, we lift g_1 to a map $g_2\colon X \to Y_\tau$ such that $f \circ g_2 = g_1$ (recall that f is L-invertible) and let $\overline{g} = \overline{h} \circ g_2$. Clearly, $\overline{g}$ is a map from X into P extending g. Therefore, $K \subset P$ is L-connected.

(4) $\Rightarrow$ (1) Take a finite subcomplex K of L. Let us first show that the pair $K \subset L$ is L-connected. Suppose Z is a space with e-dim$Z \leq L$, $A \subset Z$ closed and $g\colon A \to K$ a map such that A is normally placed in Z with respect to (g, L). Since K is C-embedded in L, A is normally placed in Z with respect to (g, K). The last condition together with the fact that K is an absolute neighborhood extensor for all separable metric spaces implies that $K \in ANE(g, A, Z)$. Indeed, we embed K in $\mathbb{R}^\omega$ and fix a retraction $r\colon U \to K$, where U is a neighborhood of K in $\mathbb{R}^\omega$. Since A is normally placed in Z with respect to (g, K), we can find a map $h\colon Z \to \mathbb{R}^\omega$ extending g. Then $h^{-1}(U)$ is a co-zero neighborhood of A in Z which contains the zero-set $h^{-1}(K)$ and $r \circ h\colon h^{-1}(U) \to K$ extends g. Hence, $K \in ANE(g, A, Z)$ which yields $L \in ANE(g, A, Z)$. Since e-dim$Z \leq L$, g can be extended to a map $\overline{g}\colon Z \to L$. Thus, $K \subset L$ is an L-connected pair. Therefore there exists a compact set $H \subset L$ containing K such that the pair $K \subset H$ is L-connected. Finally, we take a finite subcomplex P of L which contains H and observe that the pair $K \subset P$ is also L-connected. Hence, L is quasi-finite. □

Corollary 2.2. *None of the Eilenberg-MacLane complexes $K(G, n)$, $n \geq 2$ and G an Abelian group, is quasi-finite.*

Proof. This follows from Theorem 2.1(2) and the following statement (see [22, Theorem 1.4]): there exists a separable metric space X with $\dim_G X \leq 2$ and

e-dim$\beta X > L$ for every Abelian group G and every non-contractible CW-complex L. Here $\dim_G X$ denotes the cohomological dimension of X with respect to the group G. □

Let us also observe that for every quasi-finite complex L there exists a compact metrizable space which is universal for the class of all separable metric spaces of e-dim $\leq L$, in particular every space from this class has a compactification of e-dim $\leq L$. Indeed, let Y_ω be the space from Theorem 2.1(3). Then, for every X from the above class we take an embedding $i\colon X \to \mathbb{I}^\omega$ and lift i to a map $j\colon X \to Y_\omega$. The required compactification of X is the closure of $j(X)$ in Y_ω. Next corollary provides a characterization of quasi-finite countable complexes in terms of compactifications.

Corollary 2.3. *For a countable complex L the following conditions are equivalent:*

(a) *L is quasi-finite.*
(b) *For every separable metrizable space X with* e-dim$X \leq L$ *and its metrizable compactification $c(X)$ there exists a metrizable compactification $c^*(X)$ such that* e-dim$c^*(X) \leq L$ *and $c^*(X) \geq c(X)$ (i.e., there is a map from $c^*(X)$ onto $c(X)$ which is the identity on X).*

Proof. $(a) \Rightarrow (b)$ Let L be quasi-finite and X a separable metric space with e-dim$X \leq L$. We take a metric compactification $c(X)$ of X and a map $f\colon \beta X \to c(X)$ such that $f(x) = x$ for every $x \in X$. Since, by Theorem 2.1, e-dim$\beta X \leq L$, f can be factored through a metrizable compactum Z with e-dim$Z \leq L$. Clearly, Z is a compactification of X which is $\geq c(X)$.

$(b) \Rightarrow (a)$ According to [17, Corollary 3.4], there exists a metrizable compactum Y with e-dim$Y \leq L$ and a surjective map $f\colon Y \to \mathbb{I}^\omega$ such that for any map $g\colon X \to \mathbb{I}^\omega$, X being separable metrizable with e-dim$X \leq L$, there exists an embedding $i\colon X \to Y$ lifting g, i.e. $f \circ i = g$. Hence, f is L-invertible with respect to separable metric spaces. By Theorem 2.1(3), it suffices to show that f is L-invertible. Consider $g\colon Z \to \mathbb{I}^\omega$ where e-dim$Z \leq L$. According to [8, Proposition 4.9], there exist a Polish space P with e-dim$P \leq L$ and maps $h\colon Z \to P$ and $q\colon P \to \mathbb{I}^\omega$ with $g = q \circ h$. We lift q to a map $\overline{q}\colon P \to Y$ such that $f \circ \overline{q} = q$. Then $\overline{q} \circ h$ is the required lifting of g. □

Here is another property of quasi-finite complexes:

Proposition 2.4. *Every quasi-finite complex L has the following connected-pairs property:*

(CP) *For any metrizable compactum K with* e-dim$K \leq L$ *there exists a metrizable compactum P containing K such that* e-dim$P \leq L$ *and the pair $K \subset P$ is L-connected.*

Proof. Suppose K is a metrizable compactum with e-dim$K \leq L$. We embed K into the Hilbert cube $\mathbb{I}^\omega$ and take an L-invertible map $f\colon Y \to \mathbb{I}^\omega$ such that Y is a metrizable compactum with e-dim$Y \leq L$ (see Theorem 2.1(3)). Consider the adjunction space $Y \cup_f K$, i.e. the disjoint union of $Y - f^{-1}(K)$ and K with the topology consisting of the usual open subsets of $Y - f^{-1}(K)$ together with sets of the form $f^{-1}(U - K) \cup (U \cap K)$ for open subsets U of $\mathbb{I}^\omega$. There are two associated maps $p_K\colon Y \to Y \cup_f K$ and $f_K\colon Y \cup_f K \to \mathbb{I}^\omega$ such that $f = f_K \circ p_K$. Since f is L-invertible, so is f_K. Moreover, $Y - f^{-1}(K)$, being open in Y, is the union of countably many compact sets each with e-dim $\leq L$. Hence, by the countable sum theorem, e-dim$Y \cup_f K \leq L$.

We need only to show that the pair $K \subset Y \cup_f K$ is L-connected. Let $g\colon A \to K$ be a map from a closed subset $A \subset Z$ such that e-dim$Z \leq L$ and A is normally placed in Z with respect to $(g, Y \cup_f K)$. Then, considering g as a map from A into $K \subset \mathbb{I}^\omega$, we obviously have that A is normally placed in Z with respect to $(g, \mathbb{I}^\omega)$. Since $\mathbb{I}^\omega$ is an absolute extensor, there exists a map $\overline{g}\colon Z \to \mathbb{I}^\omega$ extending g. Finally, since f_K is L-invertible, we lift $\overline{g}$ to a map $h\colon Z \to Y \cup_f K$ with $f_K \circ h = \overline{g}$. Clearly, h extends g. □

Proposition 2.5. *For every $n \geq 2$ there is no $K(\mathbb{Z}, n)$-connected pair $K \subset P$ of compact sets such that K is homeomorphic to the n-dimensional sphere S^n and $\dim_{\mathbb{Z}} P \leq n$.*

Proof. We use the arguments from the proof of [17, Theorem 3.5]. Suppose for some $n \geq 2$ there is a $K(\mathbb{Z}, n)$-connected compact pair $S^n \subset P$ with $\dim_{\mathbb{Z}} P \leq n$. We choose a complex L of type $K(\mathbb{Z}, n)$ and having finite skeleta. It was shown in [18] that there exist metrizable compacta X_k, $k \geq 1$, such that:

- $\dim_{\mathbb{Z}} X_k \leq n$ for each k;
- each X_k contains a copy of S^n;
- the inclusion $i\colon S^n \hookrightarrow L$ cannot be extended over X_k so that the image of the extension is contained in the k-skeleton $L^{(k)}$ of L.

We take an extension $h\colon P \to L$ of the inclusion $i\colon S^n \hookrightarrow L$, and m such that $h(P) \subset L^{(m)}$. This means that the inclusion $j\colon S^n \hookrightarrow P$ cannot be extended to a map from X_m into P which contradicts the fact that $S^n \subset P$ is L-connected. □

The problem [27] whether, for any fixed $n \geq 2$ there is a universal space in the class of all metrizable compacta X with $\dim_{\mathbb{Z}} \leq n$ is still unsolved. Zarichnyi [28] observed that each of the above classes does not have an universal element which is an absolute extensor for the same class. Proposition 2.5 yields a little bit stronger observation.

Corollary 2.6. *None of the complexes $K(\mathbb{Z}, n)$, $n \geq 2$, have the (CP)-property.*

Recall that a map $f\colon X \to Y$ between metrizable spaces is called uniformly 0-dimensional [21] if there exists a metric on X generating its topology such that

for every $\epsilon > 0$ every point of $f(X)$ has a neighborhood U in Y with $f^{-1}(U)$ being the union of disjoint open subsets of X each of diameter $< \epsilon$. It is well known that every metric space admits uniformly 0-dimensional map into l_2.

Proposition 2.7. *Let L be a quasi-finite CW-complex. Then for every $\tau \geq \omega$ there exists a perfect L-invertible surjection $f_{(L,\tau)}\colon Y_{(L,\tau)} \to l_2(\tau)$ such that:*

(a) *$Y_{(L,\tau)}$ is a completely metrizable space of weight τ with* e-dim$Y_{(L,\tau)} \leq L$.
(b) *Every (completely) metrizable space of weight $\leq \tau$ and extension dimension $\leq L$ can be embedded as a (closed) subspace of $Y_{(L,\tau)}$.*

Proof. By Theorem 2.1(3), there exists an L-invertible map $f\colon Y \to \mathbb{I}^\omega$, where Y is a metrizable compactum with e-dim$Y \leq L$. We embed l_2 in $\mathbb{I}^\omega$ and let $Y_{(L,\omega)} = f^{-1}(l_2)$ and $f_{(L,\omega)} = f|Y_{(L,\omega)}$. Then e-dim$Y_{(L,\omega)} \leq L$ and since f is L-invertible, so is $f_{(L,\omega)}$.

If $\tau > \omega$, we take a metric d_1 on $l_2(\tau)$ and a uniformly 0-dimensional map $g\colon l_2(\tau) \to l_2$ with respect to d_1. Denote by $Y_{(L,\tau)}$ the fibered product of $l_2(\tau)$ and $Y_{(L,\omega)}$ with respect to the maps g and $f_{(L,\omega)}$. We also consider the projections $f_{(L,\tau)}\colon Y_{(L,\tau)} \to l_2(\tau)$ and $h\colon Y_{(L,\tau)} \to Y_{(L,\omega)}$. Since $f_{(L,\omega)}$ is a perfect and L-invertible surjection, so is $f_{(L,\tau)}$. If d_2 is any metric on $Y_{(L,\omega)}$, then h is uniformly 0-dimensional with respect to the metric $d = \sqrt{d_1^2 + d_2^2}$ on $Y_{(L,\tau)}$ (see [4]). Thus $Y_{(L,\tau)}$ admits a uniformly 0-dimensional map into the space $Y_{(L,\omega)}$ having extension dimension $\leq L$. Hence, by [23, Theorem 1.2], e-dim$Y_{(L,\tau)} \leq L$. Observe that $Y_{(L,\tau)}$ is completely metrizable as a perfect preimage of the completely metrizable space $l_2(\tau)$.

To prove the second item, suppose M is a metrizable space of weight $\leq \tau$ and e-dim$M \leq L$. We consider M as a subset of $l_2(\tau)$ and use the L-invertibility of $f_{(L,\tau)}$ to lift the identity map on M. Obviously this lifting is an embedding of M into $Y_{(L,\tau)}$. Moreover, if M is completely metrizable, then we can embed it in $l_2(\tau)$ as a closed subspace. This implies that the corresponding embedding of M in $Y_{(L,\tau)}$ is also closed. $\square$

A completion theorem for L-dimensional metric spaces, where L is any countable CW-complex, was established in [26]. It follows from Proposition 2.7 that this is also true for quasi-finite (not necessarily countable) complexes L.

Corollary 2.8. *Let L be a quasi-finite complex. Then every metrizable space X with* e-dim$X \leq L$ *has a completion with extension dimension $\leq L$.*

Corollary 2.9. *Let L be a quasi-finite complex and X a metrizable space. Then* e-dim$X \leq L$ *if and only if X admits a uniformly 0-dimensional map into a separable metrizable space of extension dimension $\leq L$.*

Proof. In one direction (sufficiency) this follows from the mentioned above result of Levin [23, Theorem 1.2]. Suppose X is a metrizable space of weight τ

with e-dim$X \leq L$. By Proposition 2.7, X can be embedded in the space $Y_{(L,\tau)}$. It follows from the construction of $Y_{(L,\tau)}$ that the map $h\colon Y_{(L,\tau)} \to Y_{(L,\omega)}$ is uniformly 0-dimensional. Then the restriction $h|X$ is also uniformly 0-dimensional which completes the proof. $\square$

A general factorization theorem for L-dimensional compact spaces, where L is an arbitrary complex, was proved in [24]. We provide here a factorization theorem for L-dimensional metrizable spaces with L being quasi-finite (see [23, Theorem 1.5] for similar result with L countable).

Proposition 2.10. *Let L be a quasi-finite complex and let $f\colon X \to Y$ be a map with Y metrizable. If* e-dim$X \leq L$*, then f factors through a metrizable space Z such that* e-dim$Z \leq L$ *and $w(Z) \leq w(Y)$.*

Proof. Let us first show how to reduce this proposition to the case Y is separable. This reduction is well known (see, for example, [4]), but we present it here for the reader's convenience. Suppose the result holds when the range space is separable and metrizable. We take a uniformly 0-dimensional map $g\colon Y \to l_2$ and apply the "separable factorization theorem" to the map $g \circ f\colon X \to l_2$ to obtain a separable metrizable space M and maps $q\colon X \to M$ and $h\colon M \to l_2$ with e-dim$M \leq L$ and $h \circ q = g \circ f$. Let $p_M\colon Z \to M$ and $p_Y\colon Z \to Y$ be the pullbacks of g and h respectively. Clearly, Z is a metrizable space of weight $w(Z) \leq w(Y)$. Since g is uniformly 0-dimensional, so is p_M. Then, by [23, Theorem 1.2], e-dim$Z \leq L$.

Now we prove the "separable case". Let $\tilde{Y}$ be a metrizable compactification of Y and $\tilde{f}\colon \beta X \to \tilde{Y}$ be the Čech-Stone extension of f. Since L is quasi-finite, e-dim$\beta X \leq L$. Therefore we can apply the factorization theorem of Levin-Rubin-Schapiro [24] to obtain a metrizable compactum $\tilde{Z}$ and maps $\tilde{f}_1\colon \beta X \to \tilde{Z}$ and $\tilde{f}_2\colon \tilde{Z} \to \tilde{Y}$ such that $\tilde{f}_2 \circ \tilde{f}_1 = \tilde{f}$ and e-dim$\tilde{Z} \leq L$. Then the space $Z = \tilde{f}_1(X)$ and the maps $f_1 = \tilde{f}_1|X$ and $f_2 = \tilde{f}_2|Z$ form the required factorization. $\square$

We say that a map $f\colon X \to Y$ is L-soft, where L is a CW-complex, if for any space Z with e-dim$Z \leq L$, any closed set $A \subset Z$ and any two maps $h\colon Z \to Y$ and $g\colon A \to X$, where A is normally placed in Z with respect to (g, X) and $f \circ g = h|A$, there exists a map $\overline{g}\colon Z \to X$ extending g such that $f \circ \overline{g} = h$. If, in the above definition, we additionally require Z to be from a given class of spaces $\mathcal{A}$, then we say that f is L-soft with respect to the class $\mathcal{A}$. It was established in [11] that for every countable complex L and every metric space Y there exists an L-soft map $f\colon X \to Y$ such that X is a metric space of extension dimension $\leq L$ and $w(X) = w(Y)$. We are going to show that quasi-finite complexes also have this property.

Proposition 2.11. *Let L be a quasi-finite CW-complex. Then for every $\tau \geq \omega$ there exists an L-soft map $p_{(L,\tau)}\colon X_{(L,\tau)} \to l_2(\tau)$ such that:*

(a) *$X_{(L,\tau)}$ is a completely metrizable space of weight τ with* e-dim$X_{(L,\tau)} \leq L$.
(b) *$X_{(L,\tau)}$ is an absolute extensor for all metrizable spaces of* e-dim $\leq L$.
(c) *$p_{(L,\tau)}$ is a strongly (L,τ)-universal map, i.e. for any open cover $\mathcal{U}$ of $X_{(L,\tau)}$, any (complete) metrizable space Z of weight $\leq \tau$ with* e-dim$Z \leq L$ *and any map $g\colon Z \to X_{(L,\tau)}$ there exists a (closed) embedding $h\colon Z \to X_{(L,\tau)}$ which is $\mathcal{U}$-close to g and $p_{(L,\tau)} \circ g = p_{(L,\tau)} \circ h$.*

Proof. Using Proposition 2.11 and following Zarichnyi's idea from [28] (see also [8]) that invertibility generates softness, we can show the existence of a complete separable metrizable space X with e-dim$X \leq L$ and an L-soft map $f\colon X \to l_2$. Then, as in [11], we construct the space $X_{(L,\tau)}$ and the map $p_{(L,\tau)}\colon X_{(L,\tau)} \to l_2(\tau)$ possessing the desired properties. □

3. Some more properties of quasi-finite complexes

In this section, all spaces and all CW-complexes, unless stated otherwise, are, respectively, metrizable and quasi-finite. We are going to show that some properties of finitely dominated complexes remain valid for quasi-finite complexes. We say that a space X is an absolute (neighborhood) extensor in dimension L (notation $X \in A(N)E(L)$) if for every space Z of extension dimension $\leq L$ and every map $g\colon A \to X$, where A is a closed subset of Z, there exists an extension of g over Z (resp., over a neighborhood of A in Z).

Everywhere below $cov(X)$ denotes the family of all open covers of X. Two maps $f_0, f_1\colon X \to Y$ are L-homotopic [10] if for any map $h\colon Z \to X \times [0,1]$, where Z is a space with e-dim$Z \leq L$, the composition $(f_0 \oplus f_1) \circ h|\big(h^{-1}\big(X \times \{0,1\}\big)\big)\colon h^{-1}\big(X \times \{0,1\}\big) \to Y$ admits an extension $H\colon Z \to Y$. If $\mathcal{U} \in cov(X)$ and the extension H in the above definition can be chosen such that the collection $\{H\big(h^{-1}(\{x\} \times [0,1])\big) : x \in X\}$ refines $\mathcal{U}$, then f_0 and f_1 are called $(\mathcal{U}, L)$-homotopic.

The following three propositions were given in [10] for finitely dominated countable complexes L and Polish $ANE(L)$-spaces X. Because of Proposition 2.7, one can show they also hold for quasi-finite complexes L and arbitrary (not necessarily Polish) $ANE(L)$-spaces.

Proposition 3.1. *Let X be an $ANE(L)$-space and $\mathcal{U} \in cov(X)$. Then there exists a cover $\mathcal{V} \in cov(X)$ such that any two $\mathcal{V}$-close maps of any space into X are $(\mathcal{U}, L)$-homotopic.*

Proposition 3.2. *Let $X \in ANE(L)$ and $\mathcal{U} \in cov(X)$. Then there exists a cover $\mathcal{V} \in cov(X)$ refining $\mathcal{U}$, such that the following condition holds:*

(H) *For any space Z with* e-dim$Z \leq L$, *any closed $A \subset Z$, and any two $\mathcal{V}$-close maps $f, g\colon A \to X$ such that f has an extension $F\colon Z \to X$,*

it follows that g also can be extended to a map $G\colon Z \to X$ which is $(\mathcal{U}, L)$-homotopic to F.

Proposition 3.3. *Let $X \in ANE(L)$, Z be a space with* e-dim$Z \leq L$ *and $A \subset Z$ closed. If $f, g\colon A \to X$ are L-homotopic and f admits an extension $F\colon Z \to X$, then g also admits an extension $G\colon Z \to X$, and we may be assume that F and G are L-homotopic.*

A pair of closed subsets $X_0 \subset X_1$ of a space X is called $UV(L)$-connected in X if every neighborhood U of X_1 in X contains a neighborhood V of X_0 such that $V \subset U$ is L-connected with respect to metrizable spaces, i.e. any map $g\colon A \to V$, where A is a closed subset of a space Z with e-dim $\leq L$, admits an extension $\overline{g}\colon Z \to U$. When $X_0 \subset X_0$ is $UV(L)$-connected in X, we say that X_0 is $UV(L)$ in X. If in the above definition all pairs under consideration are L-connected with respect to a given class $\mathcal{A}$, we obtain the notion of $UV(L)$-sets with respect to $\mathcal{A}$. If instead of L-connectedness of the pair $V \subset U$ we require the inclusion $V \subset U$ to be L-homotopic to a constant map in U then the pair $X_0 \subset X_1$ (resp. the set X_0) is called $UV(L)$-homotopic in X. Obviously, every $UV(L)$-connected pair is $UV(L)$-homotopic. Next corollary, which follows from Proposition 3.3, shows that these two properties are equivalent in case $X \in ANE(L)$.

Corollary 3.4. *Let X be an $ANE(L)$-space. A pair $X_0 \subset X_1$ of closed subsets of X is $UV(L)$-connected in X if and only if it is $UV(L)$-homotopic in X.*

Lemma 3.5. *Let $X_0 \subset X_1 \subset X \subset E$, where both X and E are $ANE(L)$-spaces and $X \subset E$ is closed. Then the pair $X_0 \subset X_1$ is $UV(L)$-connected in X if and only if it is $UV(L)$-connected in E.*

Proof. By Proposition 2.7, there exists a perfect L-invertible surjection $f\colon \tilde{E} \to E$ with e-dim$\tilde{E} \leq L$, and let $\tilde{X} = f^{-1}(X)$. Since $X \in ANE(L)$, we can extend $f|\tilde{X}$ to a map $g\colon W \to X$ with W being a neighborhood of $\tilde{X}$ in $\tilde{E}$. Since f is closed, we may assume that $W = f^{-1}(G)$ for some neighborhood G of X in E. The claim below follows from our constructions.

Claim. For every open $O \subset X$ the set $O^* = G - f\big(g^{-1}(X - O)\big)$ is open in G and has the following two properties: $O^* \cap X = O$ and $g\big(f^{-1}(O^*)\big) = O$.

Suppose $X_0 \subset X_1$ is $UV(L)$-connected in X. We are going to show that this pair is $UV(L)$-connected in E. To this end, let $U \subset G$ be a neighborhood of X_1 in E. Then there is a neighborhood O of X_0 in X such that $O \subset U \cap X$ is L-connected. Since U is an $ANE(L)$ (as an open subset of E), we can apply Proposition 3.2 for the space U and the one-element cover $\mathcal{U} = \{U\}$ to find an open cover $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ of U satisfying the condition (H). For every α let $G_\alpha = V_\alpha \cap \big(V_\alpha \cap X\big)^* \cap O^*$ and $V = \bigcup\{G_\alpha : \alpha \in \Lambda\}$. Obviously, $V \subset U$ is open and contains X_0. The pair $V \subset U$ is L-connected. Indeed, let $h\colon A \to V$

be a map, where $A \subset Z$ is closed and e-dim$Z \leq L$. Since f is L-invertible, h admits a lifting $h_1\colon A \to f^{-1}(V)$, i.e. $h = f \circ h_1$. According to the Claim, $g\big(f^{-1}(G_\alpha)\big) \subset V_\alpha \cap X$, $\alpha \in \Lambda$, and $V \cap X \subset O$. This implies that h and the map $h_2 = g \circ h_1\colon A \to V \cap X$ are $\mathcal{V}$-close. Since the pair $O \subset U \cap X$ is L-connected, h_2 can be extended to a map from Z into $U \cap X$. This yields, according to Proposition 3.2, that h also can be extended to a map from Z into U.

Now, suppose the pair $X_0 \subset X_1$ is $UV(L)$-connected in E. To show this pair is $UV(L)$-connected in X, let U be a neighborhood of X_1 in X. Then $U^* \subset G$ is open in E, and we can find a neighborhood V of X_0 in E such that $V \subset U^*$ is L-connected. The pair $V \cap X \subset U$ is L-connected. Indeed, any map $h\colon A \to V \cap X$, where $A \subset Z$ is closed and e-dim$Z \leq L$, admits an extension $h_1\colon Z \to U^*$. Then the map $\overline{h} = g \circ h_2\colon Z \to U$, where $h_2\colon Z \to f^{-1}(U^*)$ is a lifting of h_1, extends h. □

Theorem 3.6. *Suppose X is an $ANE(L)$-space and the pair $X_0 \subset X_1$ is $UV(L)$-connected in X. Then it is $UV(L)$-connected in any $ANE(L)$-space in which X_1 is embeddable as a closed subspace.*

Proof. Let $i\colon X_1 \to Y$ be a closed embedding, where $Y \in ANE(L)$, and M be the space obtained from the disjoint union $X \uplus Y$ by identifying all pairs of points $x \in X_1 \subset X$ and $i(x) \in Y$. The space M is metrizable and if $p\colon X \uplus Y \to M$ is the quotient map, then $p(X)$, $p(Y)$ and $p(X_1)$ are closed sets in M homeomorphic, respectively, to X, Y and X_1. Moreover, $p(X_1)$ is the common part of $p(X)$ and $p(Y)$. We embed M in a normed space E as a closed subspace. Every normed space is an absolute extensor for the class of metrizable spaces, so $E \in ANE(L)$. Since the pair $p(X_0) \subset p(X_1)$ is $UV(L)$-connected in $p(X)$, by Lemma 3.5 it is also $UV(L)$-connected in E. This implies, again by Lemma 3.5, that $p(X_0) \subset p(X_1)$ is $UV(L)$-connected in $p(Y)$. □

Corollary 3.7. *If a space X is $UV(L)$ in a given $ANE(L)$-space, then X is $UV(L)$ in any $ANE(L)$-space in which X is embeddable as a closed subset.*

In the existing literature, the UV^n- property, and more general, the $UV(L)$-property, is defined for compact spaces, see [10] and [6]. We extend this definition to arbitrary (metrizable) spaces: X is a *$UV(L)$-space* if it is $UV(L)$ in some $ANE(L)$-space containing X as a closed subspace. According to Corollary 3.7, the $UV(L)$-property does not depend on the embeddings in $ANE(L)$-spaces (for compact spaces and finite complexes L this was done in [6]). It follows from Corollary 3.4 that X is a $UV(L)$-space if and only if X is $UV(L)$-homotopic in every space $Y \in ANE(L)$ containing X as a closed subset.

Recall that a normal space X is a C-space [1] if for any sequence $\{\omega_n\}$ of open covers of X there exists a sequence $\{\gamma_n\}$ of open disjoint families such that each γ_n refines ω_n and $\cup\gamma_n$ covers X. Every finite-dimensional paracompactum, as well as every countable-dimensional metrizable space has property C [19].

We say that a complex L (not necessarily quasi-finite) possesses *the soft map property* if for every space X there exists a space Y with e-dim$Y \leq L$ and an L-soft map from Y onto X. Every countable complex has the soft map property (see [11]), as well as every quasi-finite complex (by Proposition 2.11).

A pair of spaces $\tilde{V} \subset \tilde{U}$ is called an L-extension of the pair $V \subset U$ [7] if $\tilde{U} \in AE(L)$ and there exists a map $q\colon \tilde{U} \to U$ such that the restriction $q|\tilde{V}$ is an L-soft map onto V. The following property of L-extension pairs was established in [7].

Lemma 3.8. *Let L be a complex (not necessarily quasi-finite) with the soft map property and $\tilde{V} \subset \tilde{U}$ an L-extension of the pair $V \subset U$. Let also $A \subset B$ be a pair of closed subsets of a space X with* e-dim$X \leq L$. *Suppose we have maps $f\colon B \to U$ and $g\colon A \to \tilde{U}$ such that $q \circ g = f|A$ and $f\left(\overline{B \backslash A}\right) \subset V$. Then there exists a map $h\colon X \to \tilde{U}$ such that $q \circ (h|B) = f$.*

Lemma 3.9. *Let L be a complex (not necessarily quasi-finite). Every L-connected pair $V \subset U$ of spaces admits an L-extension provided L has the soft map property.*

Proof. We take a normed space E containing V as a closed subspace and an L-soft surjection $g\colon \tilde{U} \to E$ such that $\tilde{U}$ is a space of e-dim $\leq L$. Since $V \subset U$ is L-connected, there exists a map $q\colon \tilde{U} \to U$ extending the map $g|\tilde{V}$, where $\tilde{V} = g^{-1}(V)$. Moreover, $\tilde{U} \in AE(L)$ because E is an absolute extensor for the class of metrizable spaces and g is L-soft. Therefore, $\tilde{V} \subset \tilde{U}$ is an L-extension of $V \subset U$. $\square$

If A is a subset of a space X we denote the star of A with respect to a cover $\omega \in cov(X)$ by $\mathrm{St}(A, \omega)$. We say that $\nu \in cov(X)$ is a strong star-refinement of $\omega \in cov(X)$ if for each $V \in \nu$ there exists $W \in \omega$ such that $\mathrm{St}(V, \nu) \subset W$.

Auxiliary Construction. Suppose we are given the spaces X, Z and the map $g\colon A \to X$, where $A \subset Z$ is closed. Let $\alpha_n = \{U_n(x) : x \in X\}$, $\beta_n = \{V_n(x) : x \in X\}$, $n \geq 0$, be two sequences of open covers of X and μ_n^*, $n \geq 1$, be a sequence of disjoint open families in A such that:

(1) α_n is a strong star refinement of β_{n-1} for any $n \geq 1$.
(2) each μ_n^*, $n \geq 1$, refines $g^{-1}(\beta_n)$ and $\cup\{\mu_n^* : n \geq 1\}$ is a locally finite cover of A.

We are going first to construct open and disjoint families μ_n, $n \geq 1$, in Z satisfying the following condition:

(3) $\mu = \cup\{\mu_n : n \geq 1\}$ is locally finite in Z and the restriction of each μ_n on A is μ_n^*.

To this end, we choose an upper semi-continuous (br., u.s.c.) set-valued map $r\colon Z \to A$ such that each $r(z)$ is a finite set and $r(z) = \{z\}$ for $z \in A$ (see [25] for the existence of such r). Recall that r is upper semi-continuous means that

$r^\sharp(T) = \{z \in Z : r(z) \subset T\}$ is open in Z whenever T is open in A. Obviously, $r^\sharp(T) \cap A = T$ and $r^\sharp(T_1) \cap r^\sharp(T_2) \neq \emptyset$ if and only if $T_1 \cap T_2 \neq \emptyset$ for any open subsets T, T_1 and T_2 of A. Therefore all families $\mu_n = \{r^\sharp(T) : T \in \mu_n^*\}$, $n \geq 1$, are open and disjoint in Z. Since μ^* is locally finite in A and r is finite-valued, the family $\mu = \cup\{\mu_n : n \geq 1\}$ is locally finite in Z.

The second part of our construction is to find points $x_W \in X$ such that

(4) $St\big(g(W \cap A), \alpha_n\big) \subset V_{n-1}\big(x_W\big)$ for every $W \in \mu_n$ and $n \geq 1$

This can be done as follows. Since α_n is a strong star refinement of β_{n-1} and μ_n refines $g^{-1}(\beta_n)$, for every $n \geq 1$ and $W \in \mu_n$ there exist $S \in \beta_n$ and a point $x_W \in X$ such that $St\big(g(W \cap A), \alpha_n\big) \subset St\big(S, \alpha_n\big) \subset V_{n-1}\big(x_W\big)$. The auxiliary construction is completed.

Lemma 3.10. *Let L be a complex (not necessarily quasi-finite) with the soft map property and $f\colon M \to X$ be a surjection with the following property:*

(UV) *for every $x \in X$ and its neighborhood $U(x)$ in X there exists a smaller neighborhood $V(x)$ of x such that the pair $\tilde{V}(x) = f^{-1}(V(x)) \subset \tilde{U}(x) = f^{-1}(U(x))$ is L-connected with respect to the class of metrizable spaces.*

Suppose $p\colon Y \to Z$ is a surjective map with e-dim$Y \leq L$. *Then, for any $\omega \in cov(X)$ and any map $g\colon A \to X$, where A is a closed subset of Z such that either A or $g(A)$ is a C-space, there is a neighborhood G of A in Z and a map $h\colon p^{-1}(G) \to M$ with $(f \circ h)|p^{-1}(A)$ being ω-close to $g \circ p$.*

Proof. For every $x \in X$ and $n = 0, 1, 2, \dots$ we choose a point $P(x) \in f^{-1}(x)$ and neighborhoods $U_n(x)$ and $V_n(x)$ of x in X such that the cover $\alpha_0 = \{U_0(x) : x \in X\}$ refines ω, each pair $\tilde{V}_n(x) \subset \tilde{U}_n(x)$ is L-connected with respect to all metrizable spaces and the covers $\alpha_n = \{U_n(x) : x \in X\}$, $\beta_n = \{V_n(x) : x \in X\}$ satisfy condition (1) from the auxiliary construction. Since either A or $g(A)$ is a C-space, there exists a sequence of disjoint open families $\{\mu_n^* : n \geq 1\}$ in A satisfying condition (2) above. Therefore, according to the auxiliary construction, we can extend each μ_n^* to a disjoint open family μ_n in Z such that $\mu = \cup\{\mu_n : n \geq 1\}$ is locally finite in Z and let G be the union of all elements of μ.

We introduce the following notations: $B = p^{-1}(A)$, $\overline{g} = g \circ (p|B)$, $\Omega = p^{-1}(G)$, and $\nu_n = p^{-1}(\mu_n)$. Obviously, each ν_n is a disjoint open family in Y and $\nu = \cup\{\nu_n : n \geq 1\}$ is a locally finite cover of Ω. Let us also consider the open covers $\tilde{\omega} = f^{-1}(\omega)$, $\tilde{\alpha}_n = \{\tilde{U}_n(x) : x \in X\}$ and $\tilde{\beta}_n = \{\tilde{V}_n(x) : x \in X\}$ of M corresponding, respectively, to ω, α_n and β_n. According to Lemma 3.9, every pair $\tilde{V}_n(x) \subset \tilde{U}_n(x)$ has an L-extension $\tilde{\tilde{V}}_n(x) \subset \tilde{\tilde{U}}_n(x)$ with a corresponding map $q_{n,x}\colon \tilde{\tilde{U}}_n(x) \to \tilde{U}_n(x)$ such that $\big(q_{n,x}\big)|\tilde{\tilde{V}}_n(x)$ is an L-soft surjection onto $\tilde{V}_n(x)$.

Consider the nerve $\Re$ of ν and a barycentric map $\theta\colon \Omega \to |\Re|$. Any simplex $\sigma =< W_0, W_1, .., W_k >$ from $\Re$, where $W_i \in \nu_{n(i)}$, can be ordered such that $n(0) < n(1) < ... < n(k)$. This is possible because $\cap\{W_i : i = 0, 1, .., k\} \neq \emptyset$, so the numbers $n(i)$ are different. It is easily seen that, for fixed $k \geq 1$ and $W \in \nu_k$, condition (4) from the auxiliary construction implies the following one

(5) $St\big(\overline{g}(W \cap B), \alpha_k\big) \subset V_{k-1}\big(x_W\big)$, and therefore $St\big(f^{-1}\big(\overline{g}(W \cap B)\big), \tilde{\alpha}_k\big) \subset \tilde{V}_{k-1}\big(x_W\big)$.

Let $\Sigma(\sigma)$, $\sigma \in \Re$, be the closed subset $\theta^{-1}(\sigma)$ of Ω and $\Sigma^k = \theta^{-1}(\Re^k)$, where $\Re^k$ denotes the k-th skeleton of $\Re$. For every $k \geq 0$ and $\sigma =< W_0, W_1, ..., W_k >\in \Re^k$ with $W_0 \in \nu_{n(0)}$, we define by induction maps $h_k\colon \Sigma^k \to M$ and $h_\sigma\colon \Sigma(\sigma) \to \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$ such that

(6) $h_k|\Sigma^{k-1} = h_{k-1}$ for $k \geq 1$ and $h_k|\Sigma(\sigma) = q_{n(0)-1,x_{W_0}} \circ \big(h_\sigma|\Sigma(\sigma)\big)$ for $k \geq 0$

and

(7) $f^{-1}\big(\overline{g}(W_0 \cap B)\big) \bigcup h_k\big(\Sigma(\sigma)\big) \subset \tilde{U}_{n(0)-1}(x_{W_0})$, $k \geq 0$.

We also require that

(8) $h_{\sigma_1}|\big(\Sigma(\sigma_1) \cap \Sigma(\sigma_2)\big) = h_{\sigma_2}|\big(\Sigma(\sigma_1) \cap \Sigma(\sigma_2)\big)$ for any σ_1 and σ_2 from $\Re^k$ having the same first vertex.

For $k = 0$ we define $h_0\colon \Sigma^0 \to M$ and $h_{<W>}\colon \Sigma(< W >) \to \tilde{\tilde{U}}_{n-1}(x_W)$ by $h_0\big(\Sigma(< W >)\big) = P(x_W)$ and $h_{<W>}\big(\Sigma(< W >)\big) = Q(x_W)$, where $W \in \nu_n$ and $Q(x_W)$ is a point from $\tilde{\tilde{V}}_{n-1}(x_W)$ with $q_{0,x_W}\big(Q(x_W)\big) = P(x_W)$. Obviously, h_0 restricted on every set $W \cap \Sigma^0$ is constant, so it is continuous. Moreover, every $h_{<W>}$ is also constant satisfying condition (6), and, by (5), h_0 satisfies also (7). Note that condition (8) holds for $k = 0$.

Suppose that for some $k \geq 1$ maps $h_{k-1}\colon \Sigma^{k-1} \to M$ and $h_\sigma\colon \Sigma(\sigma) \to \tilde{\tilde{U}}_{m-1}(x_W)$ satisfying conditions (6), (7) and (8) have already been defined. Here $\sigma \in \Re^{k-1}$ and $W \in \nu_m$ is the first vertex of the simplex σ.

Now, let $\sigma =< W_0, W_1, .., W_k >\in \Re^k$ with $W_i \in \nu_{n(i)}$, $i = 0, 1, .., k$. Then $\sigma \cap \Re^{k-1}$ consists of the simplexes $\sigma_i =< W_0, .., W_{i-1}, W_{i+1}, .., W_k >$, $i = 1, 2, .., k$ and the simplex $\sigma_0 =< W_1, W_2, .., W_k >$.

Claim. $f^{-1}\big(\overline{g}(W_0 \cap B)\big) \bigcup h_{k-1}\big(\Sigma(\sigma_0)\big) \subset \tilde{V}_{n(0)-1}(x_{W_0})$ *and* $f^{-1}\big(\overline{g}(W_0 \cap B)\big) \bigcup h_{k-1}\big(\Sigma(\sigma_i)\big) \subset \tilde{U}_{n(0)-1}(x_{W_0})$ *for every* $i = 1, .., k$.

Indeed, by (7) we have $f^{-1}\big(\overline{g}(W_1 \cap B)\big) \bigcup h_{k-1}\big(\Sigma(\sigma_0)\big) \subset \tilde{U}_{n(1)-1}(x_{W_1})$. But $\overline{g}(W_1 \cap B) \cap \overline{g}(W_0 \cap B) \neq \emptyset$, and hence $f^{-1}\big(\overline{g}(W_1 \cap B)\big) \bigcup h_{k-1}\big(\Sigma(\sigma_0)\big)$ is contained in $St\big(f^{-1}\big(\overline{g}(W_0 \cap B), \tilde{\alpha}_{n(1)-1}\big)\big)$. Since $n(0) \leq n(1) - 1$, $\tilde{\alpha}_{n(1)-1}$ refines $\tilde{\alpha}_{n(0)}$. This fact and the inclusion $St\big(f^{-1}\big(\overline{g}(W_0 \cap B), \tilde{\alpha}_{n(0)}\big)\big) \subset \tilde{V}_{n(0)-1}(x_{W_0})$, which follows from (5), complete the proof of the claim for $i = 0$. Since W_0 is a vertex of each σ_i, $i = 1, 2, .., k$, the other inclusions from the claim follow directly from (7).

Consider the "boundary" $\partial\Sigma(\sigma) = \bigcup_{i=0}^{i=k}\Sigma(\sigma_i)$ of $\Sigma(\sigma)$. According to the claim, $h_{k-1}\big(\partial\Sigma(\sigma)\big) \subset \tilde{U}_{n(0)-1}(x_{W_0})$ and $h_{k-1}\big(\overline{\partial\Sigma(\sigma)\backslash\Sigma_0}\big) \subset \tilde{V}_{n(0)-1}(x_{W_0})$, where $\Sigma_0 = \bigcup_{i=1}^{i=k}\Sigma(\sigma_i)$. Since the maps $h_{\sigma_i}\colon \Sigma(\sigma_i) \to \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$, $i = 1,..,k$, satisfy condition (8), they determine a map $h_\Sigma\colon \Sigma_0 \to \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$ such that $h_{\sigma_i}|\Sigma(\sigma_i) = h_\Sigma|\Sigma(\sigma_i)$ for each i. Moreover, by (6), $q_{n(0)-1,x_{W_0}} \circ h_\Sigma = h_{k-1}|\Sigma_0$. Therefore, we can apply Lemma 3.8 for the pair $\tilde{V}_{n(0)-1}(x_{W_0}) \subset \tilde{U}_{n(0)-1}(x_{W_0})$, its L-extension $\tilde{\tilde{V}}_{n(0)-1}(x_{W_0}) \subset \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$, the sets $\Sigma_0 \subset \partial\Sigma(\sigma) \subset \Sigma(\sigma)$ and the maps h_Σ and $h_{k-1}|\partial\Sigma(\sigma)$. In this way we obtain a map $h_\sigma\colon \Sigma(\sigma) \to \tilde{\tilde{U}}_{n(0)-1}(x_{W_0})$ such that $q_{n(0)-1,x_{W_0}} \circ h_\sigma|\partial\Sigma(\sigma) = h_{k-1}|\partial\Sigma(\sigma)$. Now we define $h_k\colon \Sigma^k \to M$ by $h_k|\Sigma(\sigma) = q_{n(0)-1,x_{W_0}} \circ h_\sigma$. Obviously, h_k is continuous on every "simplex" $\Sigma(\sigma)$, $\sigma \in \Re^k$, and, since the family ν is locally finite in Ω, h_k is continuous. Moreover, h_k and h_σ satisfy conditions (6), (7) and (8), and the induction is completed.

Finally, we define $h\colon \Omega \to M$ letting $h|\Sigma^k = h_k$ for each k. Continuity of h follows from continuity of each h_k and the fact that ν is locally finite. Observe also that $(f \circ h)|p^{-1}(A)$ is ω-close to $g \circ p$ because of condition (7). □

Proposition 3.11. *Let L be a complex (not necessarily quasi-finite) with the soft map property and $f_0\colon M \to X$ be a closed map such that each fiber $f_0^{-1}(x)$, $x \in X$, is $UV(L)$-connected in M. Then for every map $g_0\colon A \to X$, where A is a closed subset of a space Z with* e-dim$Z \leq L$ *such that either A or $g_0(A)$ is a C-space, there exists a neighborhood Q of A in Z and an u.s.c map $\Psi\colon Q \to M$ such that Ψ is single-valued on $Q\backslash A$ and $f_0 \circ \Psi$ is a continuous single-valued map extending g_0.*

Proof. Our proof is based on some ideas from [2, proof of Theorem 3.1]. Let f_0 and g_0 be as in the proposition. We take sequences $\{\omega_n\} \subset cov(X)$ and $\{\gamma_n\} \subset cov(A)$, and open intervals $\{\triangle_n\}$ covering the interval $J = [0,1)$, with $0 \in \triangle_1$, such that:

- ω_{n+1} is a strong star-refinement of ω_n and γ_{n+1} is a strong star-refinement of γ_n, $n = 1,2,3,\ldots$
- $\lim \mathrm{mesh}(\omega_n) = \lim \mathrm{mesh}(\gamma_n) = 0$
- $\triangle_n \cap \triangle_m \neq \emptyset$ if and only if n and m are consecutive integers.

Then $\omega = \{\omega_n \times \triangle_n : n = 1,2,...\}$ and $\gamma = \{\gamma_n \times \triangle_n : n = 1,2,...\}$ are open covers, respectively, of $X \times J$ and $A \times J$, satisfying the following conditions:

(9_i) For every point $(x,1) \in X \times I$ and its neighborhood U in $X \times I$ there exists another neighborhood V such that $St(V,\omega) \subset U$.

(9_{ii}) For every point $(a,1) \in A \times I$ and its neighborhood U in $A \times I$ there exists another neighborhood V such that $St(V,\gamma) \subset U$.

Since f_0 is a closed map all fibers of which are $UV(L)$-connected in M, the map $f = f_0 \times id\colon M \times J \to X \times J$ has the property (UV) from Lemma 3.10.

Further, let g denote the map $g_0 \times id\colon A \times J \to X \times J$ and consider an L-soft surjection $p\colon Y \to Z \times I$, $I = [0,1]$, such that Y is a space of e-dim$Y \leq L$. We have the following diagram:

$$\begin{tikzcd}
Y \arrow[d, "p\ (L\text{-soft})"'] & & M \times J \arrow[d, "f = f_0 \times id"] \\
Z \times I \supset A \times J \arrow[rr, "g = g_0 \times id"] & & X \times J
\end{tikzcd}$$

Since the product of any metrizable C-space and J is also a C-space, either $A \times J$ or $g_0(A) \times J$ is a C-space. Following the notations from Lemma 3.10, we can apply construction of this lemma by considering the spaces $M \times J$, $X \times J$, $Z \times J$, $A \times J$ and $p^{-1}(Z \times J)$ instead of the spaces M, X, Z, A and Y, respectively. Let us also note that in our situation we take α_n and β_n, $n \geq 0$, to be open covers of $X \times J$ satisfying condition (1) from the auxiliary construction with α_0 refining ω. We also require μ_n^* to be disjoint open families in $A \times J$ satisfying condition (2) such that $\mu^* = \bigcup_{n=1}^{\infty} \mu_n^*$ is a locally finite open cover of $A \times J$ which, in addition, refines γ. Then, as in the auxiliary construction, we can extend μ_n^* to disjoint open families μ_n in $Z \times J$ by choosing an u.s.c. retraction $r\colon Z \times I \to A \times I$ such that $r(z,t) \subset A \times \{t\}$ for every $t \in I$. This can be achieved by taking an u.s.c. finite-valued retraction $r_1\colon Z \to A$ and letting $r(z,t) = r_1(z) \times \{t\}$. Observe that this special choice of r implies that $r^\sharp(T)$ is open in $Z \times I$ for every open $T \subset A \times I$ and $r^\sharp(T)$ is contained in $Z \times J$ provided $T \subset A \times J$. We also pick the points $x_W \in X \times J$, $W \in \mu$, satisfying condition (4).

According to Lemma 3.10, there exists a map $h\colon p^{-1}(G) \to M \times J$, where $G = \cup\{\Lambda : \Lambda \in \mu\}$, such that each $h_k = h|\Sigma^k$ satisfies condition (7) and $(f \circ h)|\big(p^{-1}(A \times J)\big)$ is ω-close to $g \circ p$. Now, let $H = p^{-1}\big(G \cup (A \times \{1\})\big)$ and define the set-valued map $\psi\colon H \to M \times I$ letting $\psi(y) = h(y)$ if $y \in p^{-1}(G)$ and $\psi(y) = \big(f_0^{-1}(g_0(p(y))), 1\big)$ if $y \in p^{-1}(A \times \{1\})$. Let also $\psi_1 = \pi \circ \psi\colon H \to M$, where $\pi\colon M \times I \to M$ is the projection.

Claim. The map ψ_1 is u.s.c.

Since π is continuous, it suffices to prove that ψ is u.s.c. To this end, observe that $p^{-1}(G)$ is open in H and ψ is single-valued and continuous on $p^{-1}(G)$, so that we need to show only that ψ is u.s.c. at the points of $p^{-1}(A \times \{1\})$. Let $\{y_i\} \subset H$ be a sequence converging to a point $y_0 \in p^{-1}(A \times \{1\})$ and $U_0 = V_0 \times (t, 1]$ be a neighborhood of $\psi(y_0) = \big(f_0^{-1}(g_0(p(y_0))), 1\big)$ in $M \times I$. We are going to show that $\psi(y_i) \subset U_0$ for almost all i which will complete the proof of the claim. Since f_0 is a closed map, ψ is u.s.c. on $p^{-1}(A \times \{1\})$. Therefore we can assume that $\{y_i\} \subset p^{-1}(G)$, hence $\psi(y_i) = h(y_i)$ for all i. Thus $p(y_0) = (a, 1) \in A \times \{1\}$ and $p(y_i) \in G$. Since f_0 is closed, we can find a neighborhood V of $g_0(p(y_0))$ in X with $f_0^{-1}(V) \subset V_0$. By (9_i), there exists a

neighborhood $U_1 = V_1 \times (q, 1]$ of $\big(g_0(p(y_0)), 1\big)$ in $X \times I$ such that $St(U_1, \omega) \subset U = V \times (t, 1]$. Choose a neighborhood $T(a)$ of a in A with $g_0(T(a)) \subset V_1$ and apply (9_{ii}) to find a neighborhood $S = T_1(a) \times (q^*, 1]$ of $(a, 1)$ in $A \times I$ such that $St(S, \gamma) \subset T(a) \times (q, 1]$. Then $r^\sharp(S)$ is a neighborhood of $(a, 1)$ in $Z \times I$. Since $\{p(y_i)\}$ converges to $(a, 1)$, we can assume that $\{p(y_i)\} \subset r^\sharp(S)$. It suffices to show that $f(h(y_i)) \in U$ for all i. To this end, fix i and $\Lambda_0 \in \mu_{k(0)}$ containing $p(y_i)$, where $k(0)$ is the minimal k such that $p(y_i)$ is contained in some element of μ_k. Then $\Lambda_0 = r^\sharp(\Lambda_0^*)$ for some $\Lambda_0^* \in \mu^*$ and therefore $p(y_i) \in r^\sharp(\Lambda_0^*) \cap r^\sharp(S)$. Consequently, S meets Λ_0^* and let $p(y_i^*) \in \Lambda_0^* \cap S$, where $y_i^* \in p^{-1}(\Lambda_0^*)$. On the other hand, there exists $\Gamma \in \gamma$ containing Λ_0^* (recall that μ^* refines γ). Therefore, $p(y_i^*) \in St(S, \gamma) \subset T(a) \times (q, 1]$. Since $g\big(p(y_i^*)\big) = (g_0 \times id)\big(p(y_i^*)\big)$, according to the choice of $T(a) \times (q, 1]$ we have

(10) $g\big(p(y_i^*)\big) \in U_1 = V_1 \times (q, 1]$.

Since $k(0)$ is the minimal k such that y_i is contained in some $W \in \nu_k$, according to the definition of the maps h_k and condition (7) from Lemma 3.10, we have $h(y_i) \in \tilde{U}_{k(0)-1}(x_{W_0})$, where $W_0 = p^{-1}(\Lambda_0)$. The last inclusion implies $f(h(y_i)) \in U_{k(0)-1}(x_{W_0})$. Also, condition (5) from Lemma 3.10 yields that

(11) $g\big(p(y_i^*)\big) \in g\big(p\big(W_0 \cap p^{-1}(A \times J)\big)\big) \subset V_{k(0)-1}(x_{W_0})$.

Hence, both $g\big(p(y_i^*)\big)$ and $f(h(y_i))$ are points from $U_{k(0)-1}(x_{W_0})$. But the cover $\alpha_{k(0)-1}$ refines ω, and hence $U_{k(0)-1}(x_{W_0})$ is contained in an element O of ω. Therefore, O contains $g\big(p(y_i^*)\big)$ and $f(h(y_i))$. This means, according to (10), that $f(h(y_i)) \in St(U_1, \omega)$. Finally, since $St(U_1, \omega) \subset U$, we obtain $f(h(y_i)) \in U$ which completes the proof of the claim.

Now we can finish the proof. There exists a decreasing sequence $\{Q_i\}$ of open subsets of Z and an increasing sequence of real numbers $0 = t_0 < t_1 < .. < 1$ such that $\bigcap_{i=1}^{\infty} Q_i = A$, $\lim t_i = 1$, $\overline{Q}_{i+1} \subset Q_i$ and $Q_i \times [0, t_i] \subset G$ for all i. Let $\varphi_i \colon Z \to [t_{i-1}, t_i]$, $i \geq 1$, be continuous functions such that $\varphi_i(Z \backslash Q_i) = t_{i-1}$ and $\varphi_i(z) = t_i$ for $z \in \overline{Q}_{i+1}$. Then $\varphi \colon Z \to [0, 1]$ defined by $\varphi(z) = \varphi_i(z)$ for $z \in Q_i \backslash Q_{i+1}$, $\varphi(Z \backslash Q_1) = 0$, and $\varphi(A) = 1$, is continuous. Consequently, the map $\theta \colon Q_1 \to G \cup \big(A \times \{1\}\big)$, $\theta(z) = (z, \varphi(z))$, is well defined and continuous. Moreover, $\theta(z) = (z, 1)$ for all $z \in A$. Since p is L-invertible and e-dim$Q_1 \leq L$ (as an open subset of Z), we can lift θ to a map $\overline{\theta} \colon Q_1 \to H$. Then $\Psi = \psi_1 \circ \overline{\theta} \colon Q \to M$, where $Q = Q_1$,‘ is the required map. $\square$

Theorem 3.12 below is a generalization of the well known result that if G is an u.s.c. decomposition of a metrizable space X such that each element of G is UV^n in X, then X/G is LC^n [13, Theorem 11]. The result from Theorem 3.12 was also established in [6, Corollary 7.5] for finite complexes L and proper $UV(L)$-maps between Polish spaces ($UV(L)$-maps are maps with all fibers being $UV(L)$-spaces). The version of Theorem 3.12 when L is a point is a generalization

of the well known result of Ancel [3, Theorem C.5.9]. This version was also established in [12, Proposition 3.5].

Theorem 3.12. *Let L be quasi-finite and $f\colon X \to Y$ be a closed map with all fibers being $UV(L)$-connected in X. Then Y is an $ANE(L)$ with respect to C-spaces. If, in addition, X is C^L (i.e., every map into X is L-homotopic to a constant map in X), then $Y \in AE(L)$ with respect to C-spaces.*

Proof. Let $g\colon A \to Y$ be an arbitrary map, where A is a closed subspace of a space Z with e-dim$Z \le L$, such that A is a C-space. Since L is quasi-finite, it has the soft mapping property. Therefore we can apply Proposition 3.11 to obtain a neighborhood U of A in Z and an u.s.c. map $\Psi\colon U \to X$ such that Ψ is single-valued outside A and $f \circ \Psi$ is a single-valued extension of g. Hence, $Y \in ANE(L)$ with respect to C-spaces (actually we proved that $Y \in ANE(g, A, Z)$ for arbitrary $g\colon A \to Y$, where A is a closed subspace of Z such that e-dim$Z \le L$ and A is a C-space).

Suppose now that X is C^L and let $A \subset Z$ and $g\colon A \to Y$ be as above. To show that $Y \in AE(L)$ with respect to C-spaces, we need to extend g over Z. Embedding Z as a closed subset of an $AE(L)$-space with e-dim $\le L$, we can assume that $Z \in AE(L)$. Then, as before, there exists a neighborhood U of A in Z and an u.s.c. map $\Psi\colon U \to X$ such that Ψ is single-valued outside A and $f \circ \Psi$ extends g. Take neighborhoods V_1 and V_2 of A in Z such that $\overline{V_1} \subset V_2 \subset \overline{V_2} \subset U$. Let $W = Z \backslash \overline{V_1}$ and $F = W \cap \overline{V_2}$. Since $W \cap U$ is open in the $AE(L)$-space Z, the cone Cone$(W \cap U)$ is an $AE(L)$. So, the inclusion $F \subset W \cap U$ can be extended to a map $\varphi\colon W \to$ Cone$(W \cap U)$ because F is closed in W and e-dim$W \le L$. On the other hand, since $X \in C^L$, $\Psi|\big(W \cap U\big)$ is L-homotopic to a constant map in X. Consequently, the map $\Psi|F$ can be extended to a map $h\colon W \to X$. Finally, we define the set-valued map $\theta\colon Z \to X$ by $\theta(z) = h(z)$ if $z \in Z\backslash V_2$ and $\theta(z) = \Psi(z)$ otherwise. Obviously, θ is u.s.c. and single-valued outside A. Moreover, $f \circ \theta$ is the required extension of g. □

We say that a space X is locally $ANE(L)$ if every point from X is $UV(L)$ in X. Let us mention the following corollary from Theorem 3.12.

Corollary 3.13. *Let Y be locally $ANE(L)$, where L is quasi-finite. Then $Y \in ANE(L)$ with respect to C-spaces. If, in addition, $Y \in C^L$, then $Y \in AE(L)$ with respect to C-spaces.*

Remark. We can show that if, in Corollary 3.13, the property of X to be locally $ANE(L)$ is replaced by the weaker one X to be LC^L (every $x \in X$ is $UV(L)$-homotopic in X [10]), then X is an $ANE(L)$ with respect to finite-dimensional spaces (see also [6, Theorem 4.1] for a similar result).

We know that the domain and the range of a UV^n-map between compacta are simultaneously UV^n (see, for example [5]). Here is a generalization of this

result for a subclass of quasi-finite complexes. We say that a CW complex L is a C-complex if every space of e-dim $\leq L$ is a C-space. Each complex L with $L \leq S^n$ for some n (this means that e-dim$Z \leq L$ implies $\dim Z \leq n$ for any space Z) is a C-complex, in particular every sphere S^k is such a complex. Observe that Lemma 3.10 and Proposition 3.11 remain valid for C-complexes L having the soft map property without the requirements either A or $g(A)$ (resp., $g_0(A)$) to be C-spaces. This yields that, if in Theorem 3.12 and Corollary 3.13 L is a quasi-finite C-complex, then Y is an $A(N)E(L)$.

Theorem 3.14. *Let L be a quasi-finite C-complex and $f\colon X \to Y$ a closed map with $UV(L)$-fibers. Then X is $UV(L)$ if and only if Y is.*

Proof. Let E_X be a normed space containing X as a strong Z-set. This means that $X \subset E_X$ is closed and for every $\omega \in cov(E_X)$ and every map $g\colon Q \to E_X$, where Q is an arbitrary space, there is another map $h\colon Q \to E_X$ which is ω-close to g and $\overline{h(Q)} \cap X = \emptyset$ (such space E_X can be constructed as follows: embed X as a closed subset of a normed space F and let E_X be the product $F \times l_2(\tau)$, where $w(X) \leq \tau$; then $X \times \{0\}$ is a copy of X which is a strong Z-set in E_X). Identifying each fiber of f with a point, we obtain space E_Y (equipped with the quotient topology) and let $p\colon E_X \to E_Y$ be the natural quotient map. Obviously, $p(X) \subset E_Y$ is closed and, since f is a closed map, $p(X)$ is homeomorphic to Y. And everywhere below we write Y instead of $p(X)$. Moreover, p is a closed map and E_Y is metrizable. Any fiber of p is either a point or $f^{-1}(y)$ for some $y \in Y$. Hence, p is an $UV(L)$-map. Since E_X is an absolute extensor for metrizable spaces, the fibers of p are $UV(L)$-connected in E_X. Consequently, by the modified version of Theorem 3.12 for C-complexes, $E_Y \in AE(L)$.

$X \in UV(L) \Rightarrow Y \in UV(L)$. To prove this implication, by Corollary 3.7, it suffices to show that Y is $UV(L)$ in E_Y. Let U be a neighborhood of Y in E_Y. Since X is $UV(L)$ in E_X (recall that E_X is an absolute extensor) and p is closed, there exists a neighborhood V of Y in E_Y such that the pair $p^{-1}(V) \subset p^{-1}(U)$ is L-connected. We choose a neighborhood V_1 of Y in E_Y with $\overline{V}_1 \subset V$ and show that the pair $V_1 \subset U$ is L-connected. To this end, take a space Z with e-dim$Z \leq L$ and a map $h\colon A \to V_1$ with $A \subset Z$ being closed. Since U is an $ANE(L)$, there exists $\omega \in cov(U)$ satisfying condition (H) from Proposition 3.2. Further, let $\beta \in cov(E_Y)$ be the cover $\{G \cap V : G \in \omega\} \cup \{E_Y \backslash \overline{V}_1\}$. By Lemma 3.10, there exists a map $h_1\colon A \to E_X$ such that $p \circ h_1$ is β-close to h. Obviously, $h_1(A) \subset p^{-1}(V)$ and hence there exists an extension $h_2\colon Z \to p^{-1}(U)$ of h_1. Then $p \circ h_2$ is a map from Z into U such that $(p \circ h_2)|A$ is ω-close to h. Finally, according to the choice of ω, h admits an extension $\overline{h}\colon Z \to U$.

$Y \in UV(L) \Rightarrow X \in UV(L)$. As in the previous implication, it suffices to show that X is $UV(L)$ in E_X. To this end, let U be a neighborhood of X in E_X. We can assume that $U = p^{-1}(U_0)$ for some neighborhood U_0 of Y in E_Y.

Choose neighborhoods V_0, G_0 and W_0 of Y such that $V_0 \subset \overline{V_0} \subset G_0 \subset \overline{G_0} \subset W_0 \subset \overline{W_0} \subset U_0$ and the pair $G_0 \subset W_0$ is L-connected. Denote by V, G and W, respectively, the preimages $p^{-1}(V_0)$, $p^{-1}(G_0)$ and $p^{-1}(W_0)$. We claim that the pair $V \subset U$ is L-connected. Indeed, consider a map $g_V\colon A \to V$, where A is a closed subset of a space Z with e-dim$Z \leq L$. Let $\alpha \in cov(U)$ satisfy condition (H) from Proposition 3.2 and $\alpha_1 = \{T \cap G : T \in \alpha\} \cup \{E_X \backslash \overline{V}\} \in cov(E_X)$. Since X is a strong Z-set in E_X, we can find a map $g_G\colon A \to E_X$ which is α_1-close to g_V and $\overline{g_G(A)} \cap X = \emptyset$. It is easily seen that $g_G(A) \subset G$ and g_G is α-close to g_V. The last yields (because of the choice of α) that g_V can be extended to a map from Z into U if and only if g_G has such an extension. Hence, our proof is reduced to show that g_G admits an extension from Z into U. Obviously, g_G can be considered as a map from A into G_0 such that the closure $\overline{g_G(A)}$ (this is a closure in E_Y) does not meet Y. Since $G_0 \subset W_0$ is L-connected, g_G can be extended to a map $g_W\colon Z \to W_0$. Finally, consider the cover $\gamma \in cov(E_Y)$ defined by $\gamma = \{p(T\backslash X) : T \in \alpha\} \cup \{E_Y \backslash \overline{g_G(A)}\} \cup \{E_Y \backslash \overline{W_0}\}$. According to Lemma 3.10, there exists a map $g_U\colon Z \to E_X$ such that $p \circ g_U$ is γ-close to g_W. It is easily seen that $g_U(Z) \subset U$ and $g_U|A$ is α-close to g_G. The last condition implies that g_G admits an extension from Z into U which completes our proof. $\square$

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